# THE BUILDUP OF SHOCK WAVES IN AN INHOMOGENEOUS MEDIUM

## (O KUMULIATSII UDARNYKH VOLN V NEODNORODNOI SREDE)

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Papers [1 and 2] demonstrate the existence and [3] contains a systematic examination of self-similar solutions describing the domain of focusing of spherical and cylindrical shock waves in a homogeneous gas.

The class of such solutions can be extended to include the case of an inhomogeneous cold medium of density $\rho_0 = Ar^\varkappa$ ($r$ is the distance to the focusing site) and a more general equation of state than that of a gas, i.e. when the adiabatic index $\gamma$ and compression $h$ at the front are not necessarily bound by Equation $h = (\gamma + 1)/(\gamma - 1)$. This includes the problem about the limiting law of motion of a plane shock wave during emergence onto a stellar surface [4].

We note that an equation of state with $h \neq (\gamma + 1)/(\gamma - 1)$ (as one interpolated within some range of pressures) can, for example, describe a gas with allowance for ionization or liberation of energy at the shock wave front.

An important prerequisite for the occurrence of focusing with a strong wave for any positive $\varkappa$ in an inhomogeneous medium is the assumption of a zero initial pressure ($p_0 = 0$). As was shown in [5] in connection with the problem of convergence in a gas with $p_0' \neq 0$, for $\varkappa > 2(\nu - 1)$ ($\nu = 3$, 2, and 1 for a sphere, cylinder, and plane, respectively) a convergent wave degenerates into an acoustical one.

As we know, in solutions of this type the value of the self-similarity index $k$ in the expression for the self-similar variable $\xi = btr^{-k}$ is not found by analyzing the dimensionality of the problem's parameters, but is rather obtained by numerical integration from the condition of passage of the integral curve through a singular point – something which greatly complicates the study of a broad class of solutions, e.g. for arbitrary values of $\gamma$, $h$, $\varkappa$ and $\nu$. It is therefore desirable to have on hand simpler, even if approximate, results.

Such an approximate investigation can be carried out for the wave intensity variation law, as well as for the sensitivity of focusing to nonideal initial conditions.

**1. Degree of buildup in the case of ideal focusing.** Self-similar solutions describing motions behind the shock wave front contain a characteristic which arrives at the center at the same instant as the front and has the property of $\xi$-line. We shall refer to it as a $\xi$-characteristic.

Let $p$ be the pressure, $u$ the velocity of the matter, and $c$ the speed of sound in it.

Adding to the equation for the characteristic which proceeds toward the center ($dr/dt = u - c$)

$$\frac{du}{c} - \frac{1}{\gamma}\frac{dp}{p} - \frac{(\nu-1)\,u}{u-c}\frac{dr}{r} = 0 \qquad (1.1)$$

the condition of conservation of the $\xi$-line of the dimensionless ratios $\rho/\rho_0$ and $p/\rho u^2$ we obtain Equation

$$\frac{dp}{p} - \frac{1}{(1-2\gamma^{-1}\mu_2)}\frac{d\rho_0}{\rho_0} - \frac{2\mu_2}{(1-\mu_2)(1-2\gamma^{-1}\mu_2)}\frac{(\nu-1)\,dr}{r} = 0 \qquad \left(\mu = \frac{c}{u}\right) \quad (1.2)$$

for the $\xi$-characteristic.

The solution of (1.2) for $\rho_0 = Ar^{\varkappa}$ is

$$p = Br^{\alpha}, \quad \alpha = \frac{\varkappa}{1-2\gamma^{-1}\mu_2} + \frac{2(\nu-1)\mu_2}{(1-\mu_2)(1-2\gamma^{-1}\mu_2)} \qquad (1.3)$$

Because $B$ is arbitrary, this solution is valid for any line where $\xi$ = const, including the front. The value of $\mu_2$ is here taken on the $\xi$-characteristic.

Considering $\alpha$ as a function of $\mu(\xi)$ which assumes the value of the required index for $\mu = \mu_2$ and is easily computed for $\mu = \mu_1$ (the subscript 1 refers to the front), let us expand it in a series in powers of the difference between the coordinates of the $\xi$-characteristic and front $(r_2 - r_1)$,

$$\alpha = \alpha_0 + \left(\frac{d\alpha}{d\mu}\right)_1 \left(\frac{d\mu}{d\xi}\right)_1 \left(\frac{\partial\xi}{\partial r}\right)_t (r_2 - r_1) + \ldots \qquad (1.4)$$

The difference $r_2 - r_1$ is found from the condition $r_1/D_1 = r_2/(u_2 - c)_2$ (which expresses the simultaneity of the arrival of the wave and characteristic at the center) with the use of the expansion of $u$ and $c$ in $(r_2 - r_1)$ in the neighborhood of the front.

Convergence of the series with judiciously chosen $h$, $\gamma$, and positive $\varkappa$ is so rapid that even the zeroth term $\alpha_0$ of the expansion assures good accuracy. The quantity $\alpha_0$ issues from (1.3) upon replacement of $\mu_2$ by $\mu_1 = -\sqrt{\gamma(h-1)^{-1}}$ for the front of the strong shock wave

$$\alpha_0 = \frac{\varkappa}{1 + 2/\sqrt{\gamma(h-1)}} - \frac{2(\nu-1)}{1 + 2\gamma^{-1} + (h+1)/\sqrt{\gamma(h-1)}} \qquad (1.5)$$

An estimate of the error associated with approximate Formula (1.5) for $\alpha$ in accordance with the second term in (1.4) shows that the error is maximal for a plane wave ($\nu = 1$), although even in this case, and even as $\varkappa \to \infty$, the relative error in $\alpha$ is not large.

In the cases of a sphere and a cylinder with not excessively large $\varkappa$ the linear term in the expansion in $(r_2 - r_1)$ vanishes on the three surfaces of the space $h$, $\gamma$, $\varkappa$

$$h = \frac{\gamma}{\gamma+1}, \qquad \frac{x}{\nu-1} = \frac{3-h}{[1+\sqrt{\gamma^{-1}(h-1)}]^2}$$

$$\frac{x}{\nu-1} = -\mu\frac{\gamma^2 + (2\mu^2 - 2\mu - 3)\gamma - (4\mu^3 + 2\mu^2 - 2\mu)}{(1-\mu)(\gamma - 1 - 2\mu)(\gamma + \mu^2)} \qquad \left[\mu = -\left(\frac{\gamma}{h-1}\right)^{1/2}\right]$$

Equation $h = \gamma/(\gamma+1)$ corresponds to the case where the $\xi$-characteristic and front coincide ($D = u - c$) ; the rest are the conditions of vanishing of the derivatives $(d\alpha/d\mu)_1$ and $(d\mu/d\xi)_1$ . The latter equation is the result of expanding the self-similar quantities in the neighborhood of the front in $(\xi - \xi_1)$ . To this end it is convenient to make use of gas dynamics equations in self-similar variables [6] with an approximate value of the index $k$. The quantities $\alpha$ and $k$ are related by the expression $k = 1 + \frac{1}{2}(\varkappa - \alpha)$.

For a spherical wave in a homogeneous gas with $h = (\gamma+1)/(\gamma-1)$ the approximate index is equal to the exact one for $\gamma \approx 1.54$ and $\gamma \approx 2.0$ . The small error associated with the approximate formula in the other cases is illustrated by the following table:

Table 1

| $\gamma$ | $h$ | $\varkappa$ | $\nu$ | $\alpha$ | $k$ | BIBLIOGRAPHY | $\approx\alpha$ | $\approx k$ |
|---|---|---|---|---|---|---|---|---|
| 1.2 | 11 | 0 | 3 | −0.641 | 1.3206 | [3] | −0.652 | 1.3262 |
| 1.667 | 4 | 0 | 3 | −0.905 | 1.4527 | [3] | −0.902 | 1.4509 |
| 2.0 | 3 | 0 | 3 | −0.998 | 1.4992 | [3] | −1.000 | 1.5000 |
| 2.5 | 2.33 | 0 | 3 | −1.086 | 1.5431 | [3] | −1.103 | 1.5516 |
| 3.0 | 2 | 0 | 3 | −1.143 | 1.5713 | [2] | −1.117 | 1.5885 |
| 1.4 | 6 | 0 | 2 | −0.3946 | 1.1973 | [1] | −0.3941 | 1.1971 |
| 1.667 | 4 | 3.25 | 1 | 1.85 | 1.70 | [4] | 1.71 | 1.77 |

Let us take note of some of the properties of convergent waves. From (1.3) and (1.5) it follows that the pressure of plane waves in a medium of decreasing density always diminishes, increasing in the case of a cylinder and sphere only if the density does not diminish too strongly toward the center,

$$\varkappa<\left[2(\nu-1)\left(1+\frac{2}{\sqrt{\gamma(h-1)}}\right)\right]\left[1+\frac{2}{\gamma}+\frac{h+1}{\sqrt{\gamma(h-1)}}\right]^{-1} \qquad (1.6)$$

For waves with weak compression at the front ($h \to 1$), inequality (1.6) reduces to $\varkappa < 2(\nu - 1)$, i.e. to the same expression as in [5].

The temperature at the front, whose magnitude (at constant heat capacity) is $T \sim p/h\rho_0 \sim r^{\alpha-\varkappa}$, increases all the more strongly the larger the $\varkappa$. The temperature of the mass $M$ depends on its magnitude in accordance with the law

$$T \sim M^{-\beta} \qquad (M \sim r^{\varkappa+\nu})$$

$$\beta=\frac{\varkappa-\alpha}{\varkappa+\nu}=\frac{1}{\varkappa+\nu}\left[\frac{2(\nu-1}{1+2\gamma^{-1}+(h+1)/\sqrt{\gamma(h-1)}}+\frac{\varkappa}{1+{}^1/_2/\sqrt{\gamma(h-1)}}\right]$$

The limit of the exponent $\beta$ as $\varkappa \to \infty$ is finite, and, for example, in the case $\nu = 3$, $\gamma = (h + 1)/(h - 1) = {}^5/_3$ is $\approx 1.5$ times as large as for a homogeneous medium. The exponent $\beta$ likewise describes the temperature distribution in the mass at the instant of focusing which is associated with the definite value $\xi = 0$ for all $r$.

The asymptotic temperature distribution in the mass near its center (i.e. its axis or rigid wall) after focusing is given by

$$T \sim M^{-\frac{\varkappa-(\alpha/\gamma)}{\varkappa+\nu}}$$

**2. Instability of focusing.** Solution (1.3) with an approximate value of $\alpha$ (1.5) is actually a solution of the equation for the characteristic in which $\rho$, $u$ and $c$ are expressed in terms of $p$ and $\rho_0$ from the relations on the strong shock wave. A similar equation for $p$ can also be obtained for waves of finite intensity in a matter with an arbitrary equation of state, as well as for the case where not only $\rho_0$, but also the other parameters of the medium (including those which characterize the equation of state) depend on $r$. If now $(\nu - 1)d \ln r$ is replaced by $d \ln S$, then in the same approximation the equation describes the behavior of waves in channels with a slowly varying cross section $S$, after which it can be applied not only in the approximate solution of one-dimensional shock waves, but also for the description of the behavior of individual portions of a front of arbitrary shape (it is assumed that in a small neighborhood behind the front the matter flows, as it were, along channels whose walls are normal to its surface). The assumption about the spatial coincidence of the front with the characteristic is equivalent to the assumption that the variation of quantities at the shock wave occurs only by way of changes in the channel cross section or in the state of the matter ahead of the front. For this reason approximate formula (1.5) is applicable, for example, to the problem of a brief shock considered by Zel'dovich [7], where the pressure drops as a result of the expansion of the material into space, i.e. where it is fully determined by the property of the flow behind the front. The equation obtained by assuming the coincidence of the front and characteristic is asymptotically accurate for short waves of small amplitude (sound).

This method of computing the motion of a shock wave of arbitrary intensity and shape has come into wide use in the papers of foreign authors (e.g. see survey by Chester [8], which also contains a bibliography). As far as the author knows, however, the method has never before been applied to the study of shock wave buildup in an inhomogeneous medium.

In describing the motion of a front of arbirtary shape it is convenient to use the front velocity $D$ rather than $p$ as the principal variable. After $p$ has been replaced by $\rho_0 D^2 (h-1)/h$ and $\mu_2$ by $\mu_1$, Equation (1.2) becomes

$$\frac{dD}{D} + a_\rho \frac{d\rho_0}{\rho_0} + a_s(\nu - 1)\frac{dr}{r} = 0, \qquad a_\rho = \frac{1}{1 + \sqrt{\gamma(h-1)}}$$

$$a_s = 1 + \frac{2}{\gamma} + \frac{h+1}{\sqrt{\gamma(h-1)}} \tag{2.1}$$

Its solutions with $\rho_0 \sim r^\varkappa$ is

$$D = Cr^{-(\nu-1)a_s - \varkappa a_\rho} \qquad (C = \mathrm{const}) \tag{2.2}$$

We shall limit ourselves to the consideration of small perturbations, i.e. perturbations for which $\psi$, the difference between the true position $R$ of the front and its size $r$ in the absence of perturbations, is much smaller than the characteristic wavelength. In this approximation the derivatives along the lines orthogonal to the front (the walls of the elementary channels) coincide with the derivatives with respect to $R$, whereupon Equation (2.1) describing a weakly curved front becomes

$$\frac{1}{D}\frac{\partial D}{\partial R} + a_\rho \frac{\partial \ln \rho_0}{\partial R} + a_s \frac{\partial \ln S}{\partial R} = 0 \tag{2.3}$$

Differentiation here is carried out with the other coordinates (e.g. $\theta$ and $\varphi$ for a sphere) held constant. The derivative $\partial \ln S/\partial r$ is a sum of two quantities: the first of these is $(\nu - 1)R^{-1}$ while the second, which is due to the perturbation of the front, turns out to be (we shall not indicate its derivation) $-\Delta^*\psi$, where $\Delta^*$ is a Laplacian without the term

$$\frac{\partial^2}{\partial R^2} + \frac{(\nu-1)}{R}\frac{\partial}{\partial R}$$

After replacing

$$R \text{ by } r + \psi, \qquad f(R) \text{ by } f(r) + \frac{df(r)}{dr}\psi$$

and isolating the principle part, we obtain the compact form of (2.3),

$$\frac{\partial^2\psi}{\partial r^2} - \frac{\partial}{\partial r}\left(\psi \frac{d \ln D}{dr}\right) - a_s \Delta^*\psi = 0 \tag{2.4}$$

which remains completely unchanged in the case of waves of finite amplitude and an inhomogeneous medium which is at rest and has an arbitrary equation of state. In this case $d \ln D/dr$ is determined from the solution of the corresponding one-dimensional equation of the (2.1) type, while $a_s$ is a factor appearing in front of the term $d \ln S$ in such an equation.

For spherical or merely angular perturbations of cylindrical waves, solution (2.4) resolves into harmonics, and the equation for $\eta$, i.e. for the radial part of $\psi$, is of the form

$$\frac{d^2\eta}{dr^2} - \frac{d}{dr}\left(\eta \frac{d \ln D}{dr}\right) = -\frac{n(n+1)a_s}{r^2}\eta \tag{2.5}$$

Here $n$ is the number of the harmonic. In the cylindrical case $n(n+1)$ is replaced by $n^2$.

With allowance for (2.1) and (2.2), solution (2.5) for the class of problems under consideration is of the form

$$\eta \sim \frac{1}{r^{l-1}}, \qquad l_{1-2} = \frac{(\nu-1)a_s + a_\rho\varkappa + 1 \pm \sqrt{[(\nu-1)a_s + a_\rho\varkappa + 1]^2 - 4a_s n(n+1)}}{2} \tag{2.6}$$

The decrease or unbounded increase of the amplitude/radius ratio indicates stable or unstable focusing, as the case may be. The instability condition (a positive value of the real part of at least one of the roots $l$ ) coincides with the fulfilment of one of the inequalities

$$a_s(\nu-1) + a_\rho\varkappa + 1 > 0, \qquad a_s(\nu-1) + a_\rho\varkappa + 1 < -2\sqrt{a_s n(n+1)}$$

The first inequality is also the condition of a finite focusing time, as is evident from Formula

$$\tau_f = \int_{r_0}^{0} \frac{dr}{D(r)} \sim r^{a_s(\nu-1)+a_\rho\varkappa+1}\Big|_{r_0}^{0}$$

and therefore includes all cases of any interest, as well as those involving an unlimited increase in temperature.

The degree of instability increases with $\varkappa$ . As $r \to 0$ , the increase in perturbation is determined by an index associated with a "plus" in (2.6). The largest degree of growth is associated with harmonics with $n = 1$ in an inhomogeneous medium and $n = 2$ in a homogeneous medium, where the solution of the first harmonic with $l_1 = 1$ corresponds to a simple shift of the shock front, which does not prevent the wave from focusing to a point.

A similar result, i.e. the disruption of the one-dimensional progression of the shock wave with an unlimited increase in temperature at the front, also results for plane waves.

The conclusion about the instability of the self-similar focusing of shock waves which leads to an unlimited rise in temperature concurs with the hypothesis of Zababakhin [9], whereby any process involving the unlimited buildup of energy is not realized in practice due to unsufficient initial symmetry. In the case of convergent shock waves, moreover, we see that focusing is disrupted all the more rapidly the higher the degree of temperature increase in the wave. Strictly speaking, our results prove the impossibility of the unlimited buildup of energy (in a volume or mass of matter) only for the homogeneous convergence of shock waves, while the possibility of asymmetrical focusing remains open. The stability of convergence was studied approximately, by a method requiring the applicability of the initial "channel" equation ((2.1) with $(\nu-1)d \ln r$ replaced by $d \ln S$ ) to segments of the perturbed front. The applicability of the equation to the cases of cylinder and a sphere has to some extent been proven. It is valid in the same degree when the cross section of the elementary channels (or, more properly, the addition to the principal part of $S(r)$ associated with the asymmetry) varies relatively little along a length on the order of the domain of influence preceding focusing $(r_2 - r_1)$, i.e. (2.4) affords a good description of the long-wave perturbations which determine the instability.

It is possible, however, that some of the higher harmonics to which (2.4) is not applicable do indeed increase even more rapidly. But this merely reinforces the conclusion drawn concerning instability.

The solution of Equation (2.4) for short-wave perturbations implies that they propagate along the front at the velocity $D/\sqrt{a_s}$, which does not differ greatly from the actual relief rate. We note that in a gas with $h(\gamma+1)/(\gamma-1)$ the quantity $D/\sqrt{a_s}$ coincides with the relief rate for the same value $\gamma = 1.54369...$ (the root of Equation $\gamma^4 - \gamma^3 - 2 = 0$ ) for which the second term of Equation (1.4) vanishes for $\varkappa = 0$ . Apparently, the applicability of (2.4) is somewhat broader than might seem at first inspection.

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